Doi-Hopf Modules over Weak Hopf Algebras

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Abstract

The theory of Doi-Hopf modules [7, 10] is generalized to Weak Hopf Algebras [1, 12, 2].

1 Introduction

The category ${}^C\mathcal{M}(H)_A$ of Doi-Hopf Modules over the bialgebra H was introduced in [7] and independently in [10]. It is the category of the modules over the algebra A which are also comodules over the coalgebra C and satisfy certain compatibility condition involving H. The study of ${}^C\mathcal{M}(H)_A$ turned out to be very useful: It was shown in [7, 4] that many categories investigated independently before – such as the module and comodule categories over bialgebras, the Hopf modules category [15], and the Yetter-Drinfeld category [16, 14] – are special cases of ${}^C\mathcal{M}(H)_A$. Using this observation many results known for module categories over bialgebras or Hopf algebras were generalized to this more general setting [5, 6].

In this paper we generalize the definition of Doi-Hopf modules to the case when H is a Weak Bialgebra (WBA). Our definitions are supported by the fact that many results of [10, 5, 6] remain valid in this case.

Weak Bialgebras (Weak Hopf Algebras – WHA's –) are generalizations of bialgebras (Hopf algebras) see [1, 2] and [12] (latter one using somewhat different terminology). In contrast to another direction of generalization, the quasi-Hopf algebras and weak quasi-Hopf algebras, WBA's are coassociative. Though their counit is not an algebra map, their structure is designed such a way that their (left or right) (co-) module category carries a monoidal structure [12, 3] (and some more in the WHA case [3]).

WHA's have relevance for example in describing depth 2 reducible inclusions [13].

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As the bialgebra (Hopf algebra) also the WBA (WHA) is a self-dual structure: The dual space of a finite dimensional WBA (WHA) carries naturally a WBA (WHA) structure [1, 2].

The paper is organized as follows: we define and examine the structures such as the weak Doi-Hopf datum (generalizing the Doi-Hopf datum of [7]) the weak Doi-Hopf module (generalizing the Doi-Hopf module of [7]) the weak smash product (generalizing the analogous notion of [10]) and the weak Doi-Hopf integral (generalizing definitions of [6, 8]). We illustrate these notions on the same four examples generalizing some classical examples of [7, 4].

2 The Weak Doi-Hopf Datum

In this Section H is a Weak Bialgebra (WBA) in the sense of [2] over the field k. Its unit element is denoted by $\mathbb{1}$, the product of the elements $g, h \in H$ by gh, the coproduct of $h \in H$ by $\Delta(h) = h_{(1)} \otimes h_{(2)}$ and the counit is denoted by ε.

Definition 2.1 *Let H be a WBA over the field k. The k-algebra A is a* left H-comodule algebra *if there exists a left weak coaction ρ of H on A which is also an algebra map. I.e. a map $\rho : A \to H \otimes A$ such that*

$$\begin{aligned} (\mathrm{id}_H \otimes \rho) \circ \rho &= (\Delta \otimes \mathrm{id}_A) \circ \rho & (2.1a) \\ (\mathbb{1} \otimes a)\rho(1_A) &= (\Pi^R \otimes \mathrm{id}_A) \circ \rho(a) & (2.1b) \\ \rho(ab) &= \rho(a)\rho(b) & (2.1c) \end{aligned}$$

for all $a, b \in A$. We use the standard notation $\rho(a) = a_{<-1>} \otimes a_{<0>}$ and $(\Delta \otimes \mathrm{id}_A) \circ \rho(a) = a_{<-2>} \otimes a_{<-1>} \otimes a_{<0>} = (\mathrm{id}_H \otimes \rho) \circ \rho(a)$.

The left weak coaction ρ is non-degenerate *if $(\varepsilon \otimes \mathrm{id}_A) \circ \rho = \mathrm{id}_A$ or, equivalently, $(\varepsilon \otimes \mathrm{id}_A) \circ \rho(1_A) = 1_A$. For non-degenerate left weak coactions ρ (2.1b) has an equivalent form (compare with [13]) $(\Delta \otimes \mathrm{id}_A) \circ \rho(1_A) = (\mathbb{1} \otimes \rho(1_A))(\Delta(\mathbb{1}) \otimes 1_A)$.*

Similarly, A is a right H-comodule algebra *if there exists a right weak coaction ρ of H on A which is also an algebra map. I.e. a map $\rho : A \to A \otimes H$ such that*

$$\begin{aligned} (\rho \otimes \mathrm{id}_H) \circ \rho &= (\mathrm{id}_A \otimes \Delta) \circ \rho & (2.2a) \\ \rho(1_A)(a \otimes \mathbb{1}) &= (\mathrm{id}_A \otimes \Pi^L) \circ \rho(a) & (2.2b) \\ \rho(ab) &= \rho(a)\rho(b) & (2.2c) \end{aligned}$$

for all $a, b \in A$. We also denote $\rho(a) = a_{<0>} \otimes a_{<1>}$.

The right weak coaction ρ is non-degenerate *if $(\mathrm{id}_A \otimes \varepsilon) \circ \rho = \mathrm{id}_A$ or, equivalently, if $(\mathrm{id}_A \otimes \varepsilon) \circ \rho(1_A) = 1_A$. For non-degenerate right weak coactions ρ (2.2b) has an equivalent form $(\mathrm{id}_A \otimes \Delta) \circ \rho(1_A) = (1_A \otimes \Delta(\mathbb{1}))(\rho(1_A) \otimes \mathbb{1})$.*

The dual notion to comodule algebra is the module coalgebra defined as follows: The k-coalgebra C is a right H-module coalgebra *if there exists a right weak action of H on C which is also a coalgebra map. I.e. a map $\cdot : C \times H \to C$ such that*

$$\begin{aligned} (c \cdot g) \cdot h &= c \cdot (gh) & (2.3a) \\ c \cdot \Pi^L(h) &= \varepsilon_C(c_{(1)} \cdot h)c_{(2)} & (2.3b) \\ \Delta_C(c \cdot h) &= \Delta_C(c) \cdot \Delta(h) & (2.3c) \end{aligned}$$

for all $c \in C,\ g,h \in H$.

The right weak action $\cdot$ *is* non-degenerate *if* $c \cdot \mathbb{1} = c\ \forall c \in C$ *or, equivalently, if* $\varepsilon_C(c \cdot \mathbb{1}) = \varepsilon_C(c)\ \forall c \in C$. *For non-degenerate right weak actions* $\cdot$ *(2.3b) has the equivalent reformulation as* $\varepsilon_C(c \cdot h) = \varepsilon(c \cdot \Pi^L(h))$.

Similarly, C *is a* left H-module coalgebra *if there exists a left weak action of* H *on* C *which is also a coalgebra map. I.e. a map* $\cdot : H \times C \to C$ *such that*

$$
\begin{aligned}
g \cdot (h \cdot c) &= (gh) \cdot c && \text{(2.4a)}\\
\Pi^R(h) \cdot c &= c_{(1)} \varepsilon_C(h \cdot c_{(2)}) && \text{(2.4b)}\\
\Delta_C(h \cdot c) &= \Delta(h) \cdot \Delta_C(c) && \text{(2.4c)}
\end{aligned}
$$

for all $c \in C\ g,h \in H$.

The left weak action $\cdot$ *is* non-degenerate *if* $\mathbb{1} \cdot c = c\ \forall c \in C$ *or, equivalently, if* $\varepsilon_C(\mathbb{1} \cdot c) = \varepsilon_C(c)\ \forall c \in C$. *For non-degenerate left weak actions* $\cdot$ *(2.4b) has the equivalent reformulation* $\varepsilon_C(h \cdot c) = \varepsilon(\Pi^R(h) \cdot c)$.

Notice, that in contrast to the case when H is an ordinary bialgebra the unit preserving property of ρ and the counit preserving property of $\cdot$ are not required and the form of condition (b) in each group is somewhat different from the usual one.

Definition 2.2 *A* right Weak Doi-Hopf datum *is a triple* (H, A, C), *where* H *is a WBA over* k, A *a left* H*-comodule algebra and* C *a right* H*-module coalgebra.*

A left Weak Doi-Hopf datum *is a triple* (H, A, C) *where* H *is a WBA over* k, A *a right* H*-comodule algebra and* C *a left* H*-module coalgebra.*

A (left or right) Weak Doi-Hopf datum is non-degenerate *if both the weak coaction of* H *on* A *and the weak action of* H *on* C *are non-degenerate.*

Examples:

1 Let H be a WBA over k, $A := H$ as an algebra with the coaction $\rho := \Delta$, $C := H^L$ with the coalgebra structure

$$
\begin{aligned}
\Delta_{H^L}(a^L) :&= \mathbb{1}_{(2)} a^L \otimes S(\mathbb{1}_{(1)}) \equiv \mathbb{1}_{(2)} \otimes a^L S(\mathbb{1}_{(1)})\\
\varepsilon_{H^L}(a^L) :&= \varepsilon(a^L)
\end{aligned}
$$

and the action $a^L \cdot h := \mathbb{1}_{(2)} \varepsilon(a^L h \mathbb{1}_{(1)})$ for all $a^L \in H^L, h \in H$. Then $(H, A = H, C = H^L)$ is a non-degenerate right Weak Doi-Hopf datum.

2 Let H be a WBA over k, $A := H^L$ as the subalgebra of H with the coaction $\rho := \Delta|_{H^L}$, $C := H$ as a coalgebra with the action $c \cdot h := ch$ for all $c, h \in H$. Then $(H, A = H^L, C = H)$ is a non-degenerate right Weak Doi-Hopf datum.

3 Let H be a WBA over k, $A := H$ as an algebra with the coaction $\rho := \Delta$, $C := H$ as a coalgebra with the action $c \cdot h = ch$ for all $c, h \in H$. Then $(H, A = H, C = H)$ is a non-degenerate right Weak Doi-Hopf datum.

4 Let K be a WHA over k, $H := K^{op} \otimes K$ as a bialgebra. (K^{op} is the bialgebra with the same coalgebra structure as K and the opposite algebra structure.) $A := K$ as an algebra with the coaction $\rho(a) := (S^{-1}(a_{(3)}) \otimes a_{(1)}) \otimes a_{(2)}$ for all $a \in K$, $C := K$

as a coalgebra with the action $c \cdot (a \otimes b) := acb$ for all $c \in K, (a \otimes b) \in H$. Then $(H = K^{op} \otimes K, A = K, C = K)$ is a non-degenerate right weak Doi-Hopf datum.

Let us call a (left or right) weak Doi-Hopf datum *finite dimensional* if all H, A and C are finite dimensional as k-spaces. There is a well defined notion of duality for finite dimensional weak Doi- Hopf data sending a left weak Doi-Hopf datum to a right one and vice versa:

Introduce the following notations: For any finite dimensional k- space M let $\hat{M}$ denote the dual k-space. If A is a finite dimensional algebra then by $\hat{A}$ we mean the dual space equipped with the dual coalgebra structure. Similarly, for a finite dimensional coalgebra C denote the dual algebra by $\hat{C}$ and finally for a finite dimensional bialgebra H denote the dual bialgebra by $\hat{H}$.

Proposition 2.3 *For a (non-degenerate) right weak Doi-Hopf datum (H, A, C) the triple $(\hat{H}, \hat{C}, \hat{A})$ is a (non-degenerate) left weak Doi-Hopf datum – called the dual of (H, A, C) – with*

$$\begin{aligned} \hat{\rho}(\hat{c}): &= b_i \triangleright \hat{c} \otimes \beta^i \\ \phi \cdot \hat{a}: &= (\phi \otimes \hat{a}) \circ \rho \end{aligned} \tag{2.5}$$

where $\hat{c} \in \hat{C}$, $\{b_i\}$ is any basis in H and $\{\beta^i\}$ is the dual basis in $\hat{H}$, $\phi \in \hat{H}$, $\hat{a} \in \hat{A}$ and $(h \triangleright \hat{c})(d) = \hat{c}(d \cdot h)$ for $\hat{c} \in \hat{C}, d \in C, h \in H$.

Similarly, for a (non-degenerate) left weak Doi-Hopf datum (H, A, C) the triple $(\hat{H}, \hat{C}, \hat{A})$ is a (non-degenerate) right weak Doi-Hopf datum – called the dual of (H, A, C) – with

$$\begin{aligned} \hat{\rho}(\hat{c}): &= \beta^i \otimes \hat{c} \triangleleft b_i \\ \hat{a} \cdot \phi &= (\hat{a} \otimes \phi) \circ \rho \end{aligned} \tag{2.6}$$

with the obvious notation. The above duality transformation is involutive.

Proof: The transformations (2.5) and (2.6) are obviously inverses of one other. One easily checks that (2.2a) for $(\hat{H}, \hat{C}, \hat{A})$ is equivalent to (2.3a) on (H, A, C), (2.2b) to (2.3b), (2.2c) to (2.3c), (2.4a) to (2.1a), (2.4b) to (2.1b) and (2.4c) to (2.1c).

The non-degeneracy of the weak coaction $\hat{\rho}$ of $\hat{H}$ on $\hat{C}$ is equivalent to the non-degeneracy of the weak action of H on C while the non-degeneracy of the action of $\hat{H}$ on $\hat{A}$ is equivalent to the non-degeneracy of the weak coaction ρ of H on A both in the left and right cases. ∎

3 The Weak Doi-Hopf Module

Definition 3.1 *The k-space M is a* right weak Doi-Hopf module *over the right weak Doi-Hopf datum (H, A, C) if it is a non-degenerate right A- module and a non-degenerate left C-comodule i.e. there exists an action $\cdot : M \times A \to M$ for which $m \cdot 1_A = m \; \forall m \in M$ and a coaction $\rho_M : M \to C \otimes M$ for which $(\varepsilon_C \otimes \mathrm{id}_M) \circ \rho_M = \rho_M$ such that the compatibility condition*

$$\rho_M(m \cdot a) = m_{<-1>} \cdot a_{<-1>} \otimes m_{<0>} \cdot a_{<0>} \tag{3.1}$$

holds for $\rho_M(m) \equiv m_{<-1>} \otimes m_{<0>}$.

Similarly, M *is a* left weak Doi-Hopf module *over the left Doi-Hopf datum* (H, A, C) *if it is a non-degenerate left* A*-module (with* A*-action* $\cdot$*) and a non-degenerate right* C*-comodule (with* C*-coaction* ρ_M*) such that*

$$\rho_M(a \cdot m) = a_{<0>} \cdot m_{<0>} \otimes a_{<1>} \cdot m_{<1>}. \tag{3.2}$$

The category ${}^C\mathcal{M}(H)_A$ has as objects the finite dimensional right weak Doi-Hopf modules M over the right weak Doi-Hopf datum (H, A, C) and arrows $T : M \to M'$ which intertwine both the A-actions and the C-coactions:

$$T(m \cdot a) = T(m) \cdot a \quad \rho_{M'} \circ T = (\mathrm{id}_C \otimes T) \circ \rho_M \tag{3.3}$$

for all $m \in M, a \in A$.

Similarly, ${}_A\mathcal{M}(H)^C$ is the category of finite dimensional left weak Doi-Hopf modules over the left Doi-Hopf datum (H, A, C).

Let us see what categories ${}^C\mathcal{M}(H)_A$ are in our earlier examples:

Examples:

1 ${}^C\mathcal{M}(H)_A$ is equivalent to $\mathcal{M}_{A\equiv H}$, the category of right H-modules. The equivalence functor $F : {}^C\mathcal{M}(H)_A \to \mathcal{M}_A$ is the forgetful functor.

2 ${}^C\mathcal{M}(H)_A$ is equivalent to ${}^{C\equiv H}\mathcal{M}$, the category of left H-comodules. The equivalence functor $\hat{F} : {}^C\mathcal{M}(H)_A \to^C \mathcal{M}$ is the forgetful functor.

3 ${}^C\mathcal{M}(H)_A$ is equivalent to ${}^H\mathcal{M}_H$, the category of weak Hopf modules [1, 2] over H.

4 ${}^C\mathcal{M}(H)_A$ is equivalent to $\mathcal{YD}(K^{op}_{cop})^{op}$, the category of (some twisted version of) Yetter-Drinfeld modules over H. (For its definition see the Appendix).

Proposition 3.2 *Let* (H, A, C) *be a finite dimensional right weak Doi-Hopf datum and* $(\hat{H}, \hat{C}, \hat{A})$ *its dual. Then the categories* ${}^C\mathcal{M}(H)_A$ *and* ${}_{\hat{C}}\mathcal{M}(\hat{H})^{\hat{A}}$ *are equivalent.*

Proof: Let us define the functor $D : {}^C\mathcal{M}(H)_A \to_{\hat{C}} \mathcal{M}(\hat{H})^{\hat{A}}$

$$\begin{aligned} D(M) &:= \hat{M} \text{ as a } k-\text{space} \qquad & \hat{c} \cdot \mu &:= (\hat{c} \otimes \mu) \circ \rho_M \\ & & \hat{\rho}_{\hat{M}}(\mu) &:= a_i \triangleright \mu \otimes \alpha^i \\ D(T) &:= T^t & & \end{aligned} \tag{3.4}$$

where M is an object and T an arrow in ${}^C\mathcal{M}(H)_A$, ^{t} means transposition of linear operators, $\hat{c} \in \hat{C}$, $\mu \in \hat{M}$, $(a \triangleright \mu)(m) = \mu(m \cdot a)$ for $a \in A, \mu \in \hat{M}, m \in M$, $\{a_i\}$ is a basis for A and $\{\alpha^i\}$ is the dual basis for $\hat{A}$. One checks by direct calculation that D defines an equivalence functor. ∎

Proposition 3.3 *Let* (H, A, C) *be a non-degenerate right weak Doi-Hopf datum. Then the forgetful functor* $F : {}^C\mathcal{M}(H)_A \to \mathcal{M}_A$ *has a left adjoint and* $\hat{F} : {}^C\mathcal{M}(H)_A \to {}^C\mathcal{M}$ *has a right adjoint.*

Proof: Our proof is consructive. Define $G : \mathcal{M}_A \to {}^C\mathcal{M}(H)_A$ by

$$\begin{aligned} G(M) := \quad & C \cdot 1_{A<-1>} \otimes M \cdot 1_{A<0>} \quad \text{as a } k-\text{space} \\ & (c \otimes m) \cdot a := c \cdot a_{<-1>} \otimes m \cdot a_{<0>} \\ & \rho_{G(M)} := (\Delta_C \otimes \mathrm{id}_M)|_{G(M)} \\ G(T) := \quad & (id_C \otimes T) \end{aligned} \tag{3.5}$$

for M an object and T an arrow in ${}^C\mathcal{M}(H)_A$, $a \in A, (c \otimes m) \in G(M) \subset C \otimes M$.

The fact that G is a left adjoint of F is justified by the existence of unit and counit natural homomorphisms $\rho : \mathrm{id}_{{}^C\mathcal{M}(H)_A} \to G \circ F$ and $\delta : F \circ G \to \mathrm{id}_{\mathcal{M}_A}$. Define them as

$$\begin{aligned} \rho_M : M \to G(M) \qquad & \rho_M(m) := m_{<-1>} \otimes m_{<0>} \\ \delta_M : G(M) \to M \qquad & \delta_M := (\varepsilon_C \otimes \mathrm{id}_M)|_{G(M)}. \end{aligned} \tag{3.6}$$

It is staightforward to show that $\rho_M \in (M, G(M))_{{}^C\mathcal{M}(H)_A}$, and ρ is natural. The proof of $\delta_M \in (G(M), M)_{\mathcal{M}_A}$ lies on the following

Lemma 3.4 *Let (H, A, C) be a non-degenerate right weak Doi-Hopf datum. Then for any $c \in C$ and $a \in A$*

$$(i) \qquad \Delta_C(c \cdot 1_{A<-1>}) \otimes 1_{A<0>} = c_{(1)} \otimes c_{(2)} \cdot 1_{A<-1>} \otimes 1_{A<0>} \tag{3.7}$$

$$(ii) \qquad \Pi^L(a_{<-1>}) \otimes a_{<0>} = \Pi^L(1_{A<-1>}) \otimes 1_{A<0>}a. \tag{3.8}$$

Lemma 3.4 (ii) implies $\varepsilon_C(c \cdot a_{<-1>})a_{<0>} = \varepsilon_C(c \cdot 1_{A<-1>})1_{<0>}a$ and hence $\delta_M \in (G(M), M)_{\mathcal{M}_A}$. Naturality of δ is obvious.

One can proceed the same way in the case of $\hat{F}$ using now Lemma 3.4 (i). Define $\hat{G} : {}^C\mathcal{M} \to {}^C\mathcal{M}(H)_A$ as

$$\begin{aligned} \hat{G}(M) \quad := \quad & \{\varepsilon_C(m_{<-1>} \cdot a_{<-1>})m_{<0>} \otimes a_{<0>} | m \in M, a \in A\} \text{ as a } k-\text{space} \\ & (m \otimes a) \cdot b := \varepsilon_C(m_{<-1>} \cdot a_{<-1>}b_{<-1>})m_{<0>} \otimes a_{<0>}b_{<0>} \\ & \rho_{\hat{G}(M)}(m \otimes a) := m_{<-1>} \cdot a_{<-1>} \otimes m_{<0>} \otimes a_{<0>} \\ \hat{G}(T) \quad := \quad & T \otimes \mathrm{id}_A \end{aligned} \tag{3.9}$$

for M an object and T an arrow in ${}^C\mathcal{M}(H)_A$, $(m \otimes a) \in \hat{G}(M) \subset M \otimes A, b \in A$.

The unit and counit natural homomorphisms $\hat{\rho} : \mathrm{id}_{{}^C\mathcal{M}} \to \hat{F} \circ \hat{G}$ and $\hat{\delta} : \hat{G} \circ \hat{F} \to \mathrm{id}_{{}^C\mathcal{M}(H)_A}$ can be given by

$$\begin{aligned} \hat{\rho}_M : M \to \hat{G}(M) \qquad & \hat{\rho}_M(m) := \varepsilon_C(m_{<-1>} \cdot 1_{A<-1>})m_{<0>} \otimes 1_{A<0>} \\ \hat{\delta}_M : \hat{G}(M) \to M \qquad & \hat{\delta}_M(m \otimes a) := m \cdot a \end{aligned} \tag{3.10}$$

$\blacksquare$

4 The Weak Smash Product

Definition 4.1 *For the non-degenerate right weak Doi-Hopf datum (H, A, C) define the weak smash product algebra $A\#\hat{C}$ as the k-space $1_{A<0>}A \otimes 1_{A<-1>} \triangleright \hat{C}$ equipped with the multiplication rule*

$$(a\#\hat{c})(b\#\hat{d}) := (a_{<0>}b\#\hat{c}(a_{<-1>} \triangleright \hat{d})) \tag{4.1}$$

for $(a\#\hat{c}), (b\#\hat{d}) \in A\#\hat{C}$.

One checks that (4.1) makes $A\#\hat{C}$ an associative algebra with unit element $1_{A<0>}\# 1_{A<-1>} \triangleright 1_{\hat{C}}$.

Let us see what algebras $A\#\hat{C}$ are in our earlier examples.

Examples:

1 $(A \equiv H)\#(\hat{C} \equiv \hat{H}^R)$ is isomorphic to H, the isomorphism being given by $\iota : A\#\hat{C} \to H$, $\iota := \mathrm{id}_H \otimes \varepsilon_{\hat{H}}$.

2 $(A \equiv H^L)\#(\hat{C} \equiv \hat{H})$ is isomorphic to $\hat{H}$, the isomorphism being given by $\iota : A\#\hat{C} \to \hat{H}$, $\iota := \varepsilon \otimes \mathrm{id}_{\hat{H}}$.

3 $(A \equiv H)\#(\hat{C} \equiv \hat{H})$ is isomorphic to the Weyl algebra or Heisenberg double $\hat{H}\rtimes H$ [1, 2], the isomorphism being given by $\iota : A\#\hat{C} \to \hat{H}\rtimes H$, $\iota(\mathbb{1}_{(2)}a\#(\mathbb{1}_{(1)} \rightharpoonup \phi)) := \phi a$. (In all of the examples $\rightharpoonup$ denotes the Sweedler's arrow [15].)

4 $(A \equiv K)\#(\hat{C} \equiv \hat{K})$ is isomorphic to the (twisted) Drinfel'd double $\mathcal{D}(K^{op}_{cop})^{op}$ (for its definition see the Appendix). The equivalence is given by $\iota : A\#\hat{C} \to \mathcal{D}(K^{op}_{cop})^{op}$, $\iota(\mathbb{1}_{(2)}a\#(\mathbb{1}_{(1)} \rightharpoonup \phi \leftharpoonup S^{-1}(\mathbb{1}_{(3)}))) := \mathcal{D}(\phi)\mathcal{D}(a)$.

Proposition 4.2 *Let (H, A, C) be a non-degenerate right weak Doi-Hopf datum such that C is finite dimensional as a k-space. Then the categories ${}^C\mathcal{M}(H)_A$ and $\mathcal{M}_{A\#\hat{C}}$ are isomorphic.*

Proof: We have the functor $P : {}^C\mathcal{M}(H)_A \to \mathcal{M}_{A\#\hat{C}}$

$$\begin{aligned} P(M) &:= M \text{ as a } k-\text{space} \qquad m \cdot (a\#\hat{c}) := \hat{c}(m_{<-1>})m_{<0>} \cdot a \\ P(T) &:= T \end{aligned} \tag{4.2}$$

for M an object and T an arrow in ${}^C\mathcal{M}(H)_A$, $(a\#\hat{c}) \in A\#\hat{C}, m \in M$.

If C is finite dimensional as a k-space then let $\{c_i\}$ be any basis for C and $\{\gamma^i\}$ the dual basis for $\hat{C}$ and construct the inverse functor $P' : \mathcal{M}_{A\#\hat{C}} \to {}^C\mathcal{M}(H)_A$ of P:

$$\begin{aligned} P'(M) &:= M \text{ as a } k-\text{space} \qquad && m \cdot a := m \cdot (1_{A<0>}a\#1_{A<-1>} \triangleright 1_{\hat{C}}) \\ & && \rho_M(m) := c_i \otimes m \cdot (1_{A<0>}\#1_{A<-1>} \triangleright \gamma^i) \\ P'(T) &:= T \end{aligned} \tag{4.3}$$

for M an object and T an arrow of ${}^C\mathcal{M}(H)_A$, $a \in A, m \in M$.

■

5 Integrals for Weak Doi-Hopf Data

Let (H,A,C) be a non-degenerate right weak Doi-Hopf datum where H is a weak *Hopf* algebra with antipode S, $F : {}^C\mathcal{M}(H)_A \to \mathcal{M}_A$ the forgetful functor, G its left adjoint as in Proposition 3.3. V be the k-space of the natural homomorphisms $\nu : G \circ F \to \mathrm{id}_{{}^C\mathcal{M}(H)_A}$, called the *space of integrals* for the weak Doi-Hopf datum (H,A,C). We have a straightforward generalization of Theorem 2.3 of [6]:

Theorem 5.1 *The space V is isomorphic to the space V_4:*

$$\begin{aligned} V_4 := \{\gamma : C \to (C,A)_{\mathrm{Lin}} \quad &| \quad \forall c,d \in C \quad a \in A \\ \gamma(c)(d)a \quad &= \quad a_{<0>}\gamma(c \cdot a_{<-2>})(d \cdot a_{<-1>}) \\ c_{(1)} \otimes \gamma(c_{(2)})(d) \quad &= \quad d_{(2)} \cdot \gamma(c)(d_{(1)})_{<-1>} \otimes \gamma(c)(d_{(1)})_{<0>}\}. \end{aligned} \tag{5.1}$$

Furthermore the isomorphism $f_4 : V \to V_4$ takes $\nu \in V$ to a normalized element *of V_4 i.e. to an element $\gamma \in V_4$ such that $\gamma(c_{(1)})(c_{(2)}) = \varepsilon_C(c \cdot 1_{A<-1>})1_{A<0>}$ if and only if ν is a splitting of the unit natural homomorphism $\rho : \mathrm{id}_{{}^C\mathcal{M}(H)_A} \to G \circ F$.*

The relevance of the existence of normalized elements in V_4 is discussed in [6].

Let us turn to the investigation of the space of integrals over the weak Doi-Hopf datum (H,A,C) in our earlier examples. In doing so we make the additional assumption in the Examples **2** and **3** on H and in **4** on K to be a *Frobenius* WHA. Under this additional condition we identify the space of integrals for the weak Doi-Hopf datum (H,A,C) with certain subspace of the smash product algebra $A\#\hat{C}$. Also the normalization condition is formulated as a relation in the algebra $A\#\hat{C}$.

In all of the examples r be a non-degenerate right integral in H and ρ the dual right integral [2] in $\hat{H}$.

Examples:

1 The space of Doi-Hopf integrals over (H,A,C) is isomorphic to $V_0 := \mathrm{Center} H$. Construct the isomorphism $f : V_4 \to V_0$ as

$$f(\gamma) := \gamma(\mathbb{1})(\mathbb{1}). \tag{5.2}$$

The unique normalized element of V_0 is the unit element $\mathbb{1}$ of H.

2 The space of the Doi-Hopf integrals is isomorphic to $V_0 := (\hat{H}^R)' \cap \hat{H}$, the commutant of the right subalgebra in $\hat{H}$. Let us construct the isomorphism $f : V_4 \to V_0$ as

$$[f(\gamma)](h) := \varepsilon(\gamma(r)(h)) \tag{5.3}$$

for all $h \in H$.

An element $\xi \in V_0$ is normalized if

$$\hat{S}^{-1}(\rho_{(2)})\xi\rho_{(1)} = \hat{\mathbb{1}} \tag{5.4}$$

holds in $\hat{H}$.

The space V_0 is *not* isomorphic to the space $\mathcal{I}^L(\hat{H})$ of left integrals in $\hat{H}$. It is its subspace $\hat{H}^L$ which is isomorphic to $\mathcal{I}^L(\hat{H})$ via the isomorphism $g : \mathcal{I}^L(\hat{H}) \to \hat{H}^L$,

$g(\lambda) := \hat{S}(\lambda \leftharpoonup r)$. It is but true that the existence of normalized elements in $\mathcal{I}^L(\hat{H})$ and V_0 are equivalent.

3 The space of the Doi-Hopf integrals is isomorphic to $V_0 := H' \cap (\hat{H} \rtimes H)$, the commutant of H in the Weyl algebra. The isomorphism $f : V_4 \to V_0$ is given by

$$f(\gamma) := \beta^i \gamma(r)(b_i) \tag{5.5}$$

with the help of the basis $\{b_i\}$ of H and the dual basis $\{\beta^i\}$ of $\hat{H}$.

The element $w \in V_0$ is normalized if

$$\hat{S}^{-1}(\rho_{(2)}) w \rho_{(1)} = 1_{\hat{H} \rtimes H} \tag{5.6}$$

holds in the Weyl algebra $\hat{H} \rtimes H$.

4 The space of the Doi-Hopf integrals is isomorphic to $V_0 := \{u \in \mathcal{D}(K^{op}_{cop})^{op} | \; u\mathcal{D}(b) = \mathcal{D}(b_{(1)}) u \mathcal{D}(S^{-1}(r) S^{-2}(b_{(2)}) \rightharpoonup \rho)\}$. The isomorphism $f : V_4 \to V_0$ is given by

$$f(\gamma) := \mathcal{D}(\beta^i)\mathcal{D}(\gamma(r)(b_i)) \tag{5.7}$$

with the help of the basis $\{b_i\}$ of H and the dual basis $\{\beta^i\}$ of $\hat{H}$.

$u \in V_0$ is normalized if

$$\hat{S}^{-1}(\rho_{(2)}) u \rho_{(1)} = 1_{\mathcal{D}(K^{op}_{cop})^{op}} \tag{5.8}$$

holds in the double $\mathcal{D}(K^{op}_{cop})^{op}$.

6 Appendix: Yetter-Drinfel'd modules over WHA's and Drinfel'd doubles

For the convenience of the reader we give here the generalization of the double construction due to Drinfel'd [9] and of the corresponding theory of Yetter-Drinfel'd modules [16, 14] to WHA's.

Definition 6.1 *[1] Let H be a finite dimensional WHA over the field k. Its* Drinfel'd double $\mathcal{D}(H)$ *is the WHA defined below:*

As a k-space $\mathcal{D}(H)$ is an amalgamated tensor product $H_{H^L \equiv \hat{H}^R} \otimes_{H^R \equiv \hat{H}^L} \hat{H}$ with the amalgamation relations $a^R \otimes \hat{\mathbb{1}} \equiv \mathbb{1} \otimes (\hat{\mathbb{1}} \leftharpoonup a^R)$; $(a^L \otimes \hat{\mathbb{1}}) \equiv \mathbb{1} \otimes (a^L \rightharpoonup \hat{\mathbb{1}})$ for $a^L \in H^L, a^R \in H^R$. Denote by $\mathcal{D}(a)\mathcal{D}(\phi)$ the image of $H \otimes \hat{H} \ni a \otimes \phi$ under the amalgamation and $\mathcal{D}(a) \equiv \mathcal{D}(a)\mathcal{D}(\hat{\mathbb{1}}), \mathcal{D}(\phi) \equiv \mathcal{D}(\mathbb{1})\mathcal{D}(\phi)$.

The algebra structure is defined by

$$\begin{aligned} \mathcal{D}(a)\mathcal{D}(b) &= \mathcal{D}(ab) \\ \mathcal{D}(\phi)\mathcal{D}(\psi) &= \mathcal{D}(\phi\psi) \\ \mathcal{D}(\phi)\mathcal{D}(a) &= \mathcal{D}(a_{(2)})\mathcal{D}(\phi_{(2)}) \langle \phi_{(1)} | a_{(3)} \rangle \langle \phi_{(3)} | S^{-1}(a_{(1)}) \rangle. \end{aligned} \tag{6.1}$$

One checks that (6.1) is compatible with the amalgamation relations and makes $\mathcal{D}(H)$ an associative algebra with unit $\mathcal{D}(\mathbb{1}) \equiv \mathcal{D}(\hat{\mathbb{1}})$.

The colagebra structure is given by

$$\begin{aligned} \Delta_{\mathcal{D}}(\mathcal{D}(a)\mathcal{D}(\phi)) &= \mathcal{D}(a_{(1)})\mathcal{D}(\phi_{(2)}) \otimes \mathcal{D}(a_{(2)})\mathcal{D}(\phi_{(1)}) \\ \varepsilon_{\mathcal{D}}(\mathcal{D}(a)\mathcal{D}(\phi)) &= \varepsilon(a(\phi \rightharpoonup \mathbb{1})) \equiv \hat{\varepsilon}((\hat{\mathbb{1}} \leftharpoonup a)\phi). \end{aligned} \tag{6.2}$$

One checks that (6.2) makes $\mathcal{D}(H)$ a WBA. Finally the antipode is

$$S_{\mathcal{D}}(\mathcal{D}(a)\mathcal{D}(\phi)) = \mathcal{D}(\hat{S}^{-1}(\phi))\mathcal{D}(S(a)) \tag{6.3}$$

making $\mathcal{D}(H)$ a WHA.

Definition 6.2 *Let H be a WBA over the field k. The k-space M is a right* Yetter-Drinfel'd module *over H if it is a non-degenerate right H-module and a non-degenerate left H comodule s.t.*

$$\begin{aligned} m_{<-1>}a_{(1)} \otimes m_{<0>} \cdot a_{(2)} &= a_{(2)}(m \cdot a_{(1)})_{<-1>} \otimes (m \cdot a_{(1)})_{<0>} \\ m_{<-1>}\mathbb{1}_{(1)} \otimes m_{<0>} \cdot \mathbb{1}_{(2)} &= m_{<-1>} \otimes m_{<0>} \end{aligned} \tag{6.4}$$

for all $m \in M, a \in A$.

Notice that if H is also a WHA then (6.4) can be replaced by the single relation

$$(m \cdot a)_{<-1>} \otimes (m \cdot a)_{<0>} = S^{-1}(a_{(3)})m_{<-1>}a_{(1)} \otimes m_{<0>} \cdot a_{(2)}. \tag{6.5}$$

By the category $\mathcal{YD}(H)$ we mean the category with objects the finite dimensional right Yetter-Drinfel'd modules over H and arrows $T : M \to M'$ intertwining both the H-module and the H-comodule structures of M and M'.

If H is a finite dimensional WHA then by our Proposition 4.2 and Example 4. the category $\mathcal{YD}(H)$ is equivalent to the category of the right modules over the WHA $\mathcal{D}(H)$ hence carries (among others) a monoidal structure [1, 2]. It is not so obvious however that it is true for any WBA H:

Proposition 6.3 *Let H be a WBA over the field k. Then the category $\mathcal{YD}(H)$ has a monoidal structure.*

Proof: Our proof is constructive. For two objects M, N and arrows T, S of $\mathcal{YD}(H)$ let

$$\begin{aligned} M \times N \quad &:= M \cdot \mathbb{1}_{(1)} \otimes N \cdot \mathbb{1}_{(2)} \quad \text{as a } k-\text{space} \\ &(m \otimes n) \cdot a := m \cdot a_{(1)} \otimes n \cdot a_{(2)} \\ &\rho_{M\times N}(m \otimes n) := n_{<-1>}m_{<-1>}\mathbb{1}_{(1)} \otimes m_{<0>} \cdot \mathbb{1}_{(2)} \otimes n_{<0>} \cdot \mathbb{1}_{(3)} \\ T \times S \quad &:= (T \otimes S) \circ \Delta(\mathbb{1}) \end{aligned} \tag{6.6}$$

with $m \otimes n \in M \times N, a \in A$. The monoidal unit is

$$\begin{aligned} H^L \text{ as a } k-\text{space} \qquad & a^L \cdot h := \mathbb{1}_{(2)}\varepsilon(a^L h \mathbb{1}_{(1)}) \\ & \rho_{H^L} := \Delta|_{H^L} \end{aligned} \tag{6.7}$$

for $a^L \in H^L, h \in H$.

The reader may check using some WBA calculus that all $M \times N$ and H^L are Yetter-Drinfel'd modules over H if M and N are.

In order to prove that H^L is a monoidal unit for the category $\mathcal{YD}(H)$ one has to construct the invertible intertwiners $u^L_M \in (M, H^L \times M)_{\mathcal{YD}(H)}$, $u^R_M \in (M, M \times H^L)_{\mathcal{YD}(H)}$ satisfying the triangle identities [11] and being natural in M. They are as follows:

$$\begin{aligned} u^L_M(m) &= \mathbb{1}_{(2)} \otimes m \cdot \Pi^L(\mathbb{1}_{(1)}) \\ u^R_M(m) &= m \cdot \mathbb{1}_{(1)} \otimes \mathbb{1}_{(2)}. \end{aligned} \tag{6.8}$$

for all $m \in M$ and all objects M of $\mathcal{YD}(H)$. ∎

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